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The Lyapunov Spectrum for Conformal Expanding Maps and Axiom-A Surface Diffeomorphisms

Howard Weiss[1]

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We provide a detailed description of the decomposition of a conformal repeller by the level sets of the Lyapunov exponent, along with a similar result for Axiom-A surface diffeomorphisms.

KEY WORDS: Axiom-A surface diffeomorphism; conformal repeller; dimension spectrum; expanding map; Hausdorff dimension; Lyapunov exponent; Lyapunov spectrum; multrifractal analysis; pointwise dimension.

I. INTRODUCTION

Lyapunov exponents measure the exponential rate of divergence of *infinitesimally close* orbits of a smooth dynamical system. These exponents are intimately related to the global stochastic behavior of the system and are fundamental invariants of a smooth dynamical system. In [EP], Eckmann and Procaccia suggested an analysis of Lyapunov exponents for chaotic dynamical systems. This suggestion was further investigated on a physical level by Szépfalusy and Tél [ST] and by Tél [T], but no authors have been able to provide rigorous proofs. In Sections II and III, we effect a rigorous analysis for conformal repellers and Axiom-A surface diffeomorphisms and gain new insights into the *distribution* of Lyapunov exponents, including the precise values attained by the Lyapunov exponents, the *size* and structure of the corresponding level sets, and the *size* and structure of the set of points for which the exponent does not exist.

These results are examples of a multifractal analysis *in the extended sense*. The traditional notion of multifractal analysis involves decomposing a fractal set into the level sets of the pointwise dimension. In our general

[1] Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802; e-mail: weiss@math.psu.edu.

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concept of multifractal analysis, one studies the fundamental invariants in smooth ergodic theory (including Lyapunov exponents, local entropy, Birkhoff averages, and pointwise dimension) and effects a comprehensive analysis of the complicated decomposition of the phase space into level sets of these invariants. Important elements of the analysis should include determining the precise range of values these invariants attain, a thorough analysis of the topological and dimension properties of the level sets, and an understanding of the sets where the limits do not exist.

An important tool in studying these questions is the Lyapunov spectrum, which records the Hausdorff dimension of the level sets for the Lyapunov exponent. We show that for most conformal repellers, this spectrum (map) is real analytic and strictly convex on an interval. It follows that the range of the Lyapunov exponent contains an open interval of values and hence the Lyapunov exponent attains uncountably many distinct values. For each value in this interval, we construct an equilibrium state fully supported on the corresponding level set. This implies that the level sets are dense in the repeller. We also prove similar results for *most* Axiom-A surface diffeomorphisms.

It is quite striking that while the Lyapunov exponent is intrinsically only a measurable function and the level sets in the decomposition are extremely intertwined, the Lyapunov spectrum, which encodes this very complicated decomposition, is smooth and convex.

Our strategy consists of first establishing a simple link between the Lyapunov spectrum and the dimension spectrum for the measure of maximal entropy and then using results from [PW2, PW3] on the dimension spectrum to obtain *analogous* results for the Lyapunov spectrum. These results are the first application of the multifractal analysis, currently a very popular area of research, to an object other than pointwise dimension.

One intriguing dynamical consequence is a rigidity result for rational maps which says that if the Lyapunov exponent for a hyperbolic rational map attains only countably many values, then the map must be of the form $z \to z^{\pm n}$ for some $n \in \mathbb{Z}$.

We then apply a result in [BS] and conclude that for *most* nonformal repellers, there is a dense set of *maximal* Hausdorff dimension on which the Lyapunov exponent does not exist, along with the analogous results for the positive and negative exponents for an Axiom-A surface diffeomorphism. This observation complements the results mentioned above, and together they yield, for certain classes of hyperbolic dynamical systems, a complete picture of the extremely complicated decomposition by level sets of the Lyapunov exponent.

Finally, we present a simple rigidity result for a geodesic flow on a negatively curved surface involving the.Lyapunov exponents. This is a geometric counterpart to the rigidity result mentioned above for rational maps.

II. CONFORMAL REPELLERS

Let M be a smooth manifold and $g: M \to M$ a $C^{1+\gamma}$ map for $\gamma > 0$. Suppose that J is a compact invariant subset of M ($g(J) = J$), and consider the map g restricted to J. We say that g is a *conformal expanding map* and J is *conformal repeller* if there exists a Riemannian metric on J and a function $a(x)$ such that for all $x \in J$ we have that $dg_x = a(x)\,\mathrm{Isom}(x)$, where $\mathrm{Isom}(x)$ denotes an isometry of $T_x J$, dg_x is the differential of g at x, and $|a(x)| > 1$ for all $x \in J$. Clearly $a(x) = \|dg_x\|$. Furthermore, we require that the map $g \mid J$ be a local homeomorphism. Examples of conformal expanding maps include Markov maps of an interval, rational maps with hyperbolic Julia sets, and conformal toral endomorphisms. We will follow custom and refer to g as a conformal repeller.

We define the *Lyapunov exponent* of g at x by

$$\chi(x) = \lim_{n \to \infty} \frac{1}{n} \log \|dg_x^n\| = \frac{1}{n} \log \prod_{k=0}^{n-1} |a(g^k(x))|$$

if the limit exists. If the limit exists at a point x, then $\chi(x)$ is uniformly bounded away from zero since $\chi(x) \geqslant \min_{x \in J} \log |a(x)| > 0$. Let ν be an invariant Borel probability measure for g which is supported on J. It follows from the Subadditive Ergodic Theorem that $\chi(x)$ exists on a total probability set, i.e., a set that has full measure with respect to each invariant measure. This function is measurable but typically is not continuous.

If the measure ν is ergodic, for example if ν is an equilibrium state (see Appendix) on J, then $\chi(x) = \chi_\nu = \int_J \log |a(x)|\, d\nu(x)$ for ν-almost every $x \in J$, and we obtain the decomposition of the repeller J by

$$J = \{x \in J : \chi(x) = \chi_\nu\} \cup \bigcup_{\beta \in \mathbb{R} \setminus \{\chi_\nu\}} \{x \in J : \chi(x) = \beta\}$$
$$\cup \{x \in J : \chi(x) \text{ does not exist}\}$$

We call χ_ν the *Lyapunov exponent of* ν.

There are several fundamental questions related to this decomposition. Do there exist points x such that $\chi(x)$ exists but does not equal χ_ν? Since the ν measure of this set is zero, what is the Hausdorff dimension of this set? What values are attained by $\chi(x)$? Do there exist points x such that $\chi(x)$ does not exist, and if so, what is the Hausdorff dimension of this set?

Since Lyapunov exponents are fundamental invariants of a smooth dynamical system, it seems important to have a good understanding of this decomposition.

To study related questions, Eckmann and Procaccia defined the *Lyapunov spectrum* for the map g by

$$l(\beta) = \dim_H L_\beta, \qquad \text{where} \quad L_\beta = \{x \in J : \chi(x) = \beta\}$$

and $\dim_H L_\beta$ denotes the Hausdorff dimension of the level set L_β. The Lyapunov spectrum will be a major tool in our analysis.

The second ingredient in our analysis is the dimension spectrum (for pointwise dimension). The dimension spectrum is one of the principle components in the multifractal analysis of measures on fractals. In [PW2] the authors effect a complete multifractal analysis of equilibrium states for conformal repellers and in [PW3] the authors effect a complete multifractal analysis of equilibrium states for Axiom-A surface diffeomorphisms.

Let v be an invariant Borel probability measure on J for g. Given $x \in J$ we consider the *pointwise dimension of* v *at* x,

$$d_v(x) = \lim_{r \to 0} \frac{\log v(B(x, r))}{\log r}$$

if the limit exists, where $B(x, r)$ denotes the ball of radius r centered at x. We call a measure v *exact dimensional* if there exists a number s such that $d_v(x) = s$ for v-almost every point $x \in J$. In [PW1] we show that equilibrium states for conformal repellers are exact dimensional and $d_v(x) = h_v(g)/\chi_v$, where $h_v(g)$ is the measure theoretic entropy of g.

The multifractal analysis is a description of the fine-scale geometry of the set J whose constituent components are the level sets $K_{v,\alpha} = \{x \in J : d_v(x) = \alpha\}$ for $\alpha \in \mathbb{R}$. The *dimension spectrum*, denoted by $f_v(\alpha)$, is defined by

$$f_v(\alpha) = \dim_H K_{v,\alpha}$$

We now discuss a symbolic model for the conformal repeller and a method for computing the pointwise dimension of equilibrium states using the symbolic model. It is well known that conformal repellers have Markov partitions consisting of partition elements $\mathfrak{R} = \{R_1, ..., R_m\}$ with arbitrarily small diameter such that each set R_i is the closure of its interior $\mathring{R}_i$, $J = \bigcup_i R_i$, $\mathring{R}_i \cap \mathring{R}_j = \varnothing$ unless $i = j$, and each $g(R_i)$ is a union of sets R_j [R2].

The Markov partition generates a symbolic model of g on J by a one-sided subshift of finite type (Σ_A^+, σ) where A is the incidence matrix of the

Markov partition and $\sigma: \Sigma_A^+ \to \Sigma_A^+$ is the shift map. This gives a coding map $\pi: \Sigma_A^+ \to J$ which is Hölder continuous, surjective, and injective on the set of points whose trajectories never intersect the boundary of any element of the Markov partition. With respect to this coding map, the shift map models g, i.e., $\pi \circ \sigma = g \circ \pi$. Furthermore, the cardinality of $\pi^{-1}(x)$ is uniformly bounded for all $x \in J$. The pullback by π of any Hölder continuous function on J is Hölder continuous on Σ_A^+. Furthermore, the pushforward of any Gibbs measure (see Appendix) on Σ_A^+ is an equilibrium measure on J and the pullback of any equilibrium state on J is a Gibbs measure on Σ_A^+ [Bo].

Define the *Markov balls*

$$\Delta_{i_1 \cdots i_n} = R_{i_1} \cap g^{-1} R_{i_2} \cap \cdots \cap g^{-n+1} R_{i_n}$$

where g^{-i} denotes a branch of the inverse of g^i. By the Markov property, every Markov ball has the property that $\Delta_{i_1 \cdots i_n} = R_{i_1} \cap g^{-n+1} R_{i_n}$.

The pointwise dimension is defined using metric balls. However the approach we take to study pointwise dimension essentially replaces balls by Markov balls in the definition of pointwise dimension. For a point $x \in J$ we consider the quantity

$$\lim_{n \to \infty} \frac{\log \nu(\Delta_{i_1 \cdots i_n}(x))}{\log \operatorname{diam}(\Delta_{i_1 \cdots i_n}(x))}$$

where $\Delta_{i_1 \cdots i_n}(x)$ denotes the Markov ball at level n that contains the point x. The careful reader will notice that this quantity is not well defined for points x whose orbit intersects the boundary of the Markov partition. Although the boundary has measure zero with respect to any ergodic measure, it may have positive Hausdorff dimension.

To overcome this technical difficulty, we define the analogous quantity on the symbolic model. This will give a new notion of local dimension which can rightfully be called the Markov or symbolic pointwise dimension. It is not a priori clear, and requires a nontrivial proof, that for equilibrium states these two notions of local dimension *essentially* coincide. This is the content of Theorem 2.2. We caution the reader that in the multifractal literature, a large number of authors claim to prove results about pointwise dimension but actually only prove results about Markov pointwise dimension. The main advantage of working with Markov pointwise dimension is that for equilibrium states, the measure of Markov balls can be uniformly estimated using symbolic dynamics. Also, the repeller J can be naturally viewed as a limit set for a geometric construction using these Markov balls [PW1].

Let ξ be a Hölder continuous function on J and $\nu = \nu_\xi$ the corresponding equilibrium state for g. Denote by ξ^* the pull back of ξ under the coding map π, i.e., $\xi^* = \xi \circ \pi$, and by $\mu = \mu_{\xi^*}$ the Gibbs measure corresponding to ξ^*. Given a number $\alpha \geqslant 0$ define

$$\hat{K}_{\mu, \alpha} = \left\{ \omega \in \Sigma_A^+ : \lim_{n \to \infty} \frac{\log \mu(C_n(\omega))}{\log \prod_{k=0}^{n-1} |a(\pi(\sigma^k(\omega)))|^{-1}} = \alpha \right\} \tag{1}$$

where $C_n(\omega)$ denotes the n-cylinder that contains the point ω. A routine application of the Jacobian estimate [KH] shows that if $x = \pi(\omega)$ then

$$\chi(x) = \lim_{n \to \infty} \frac{\log \prod_{k=0}^{n-1} |a(\pi(\sigma^k(\omega)))|}{n} = \lim_{n \to \infty} -\frac{1}{n} \log \operatorname{diam}(\Delta_{i_1 \cdots i_n}(x))$$

The following useful theorem says that for equilibrium states, the Markov pointwise dimension (defined for points ω in the symbolic model by (1)) coincides with the pointwise dimension (at $\pi(\omega)$) on J.

Theorem 2.1 [PW1]. Let $g: J \to J$ be a $C^{1+\gamma}$ nonformal repeller, let ν be an equilibrium state corresponding to a Hölder continuous function, and let μ be the pullback (Gibbs) measure (see Appendix) on Σ_A^+.

(1) For every $\omega \in \hat{K}_{\mu, \alpha}$ we have that $d_\nu(x) = \alpha$, where $x = \pi(\omega)$.

(2) For every $x \in K_{\nu, \alpha}$ there exists $\omega \in \hat{K}_{\mu, \alpha}$ such that $\pi(\omega) = x$.

In other words, $\pi(\hat{K}_{\mu, \alpha}) = K_{\nu, \alpha}$.

Ruelle [R1] showed that the Hausdorff dimension d of J is given by *Bowen's formula* $P(-d \log |a|) = 0$, where P is the thermodynamic pressure (see Appendix), and that the d-Hausdorff measure is equivalent to the equilibrium state ν_{dim} corresponding to the Hölder continuous potential $-d \log |a|$. The measure ν_{dim} plays a special role in the multifractal analysis and we call this measure the *measure of maximal dimension*. Let ν_{max} denote the *measure of maximal entropy* for g, i.e., the equilibrium state for a constant potential.

The following theorem is part of the multifractal analysis of equilibrium states for conformal repellers.

Theorem 2.2 [PW2]. Let $g: J \to J$ be a $C^{1+\gamma}$ conformal repeller and let $\nu = \nu_\xi$ be the equilibrium state corresponding to the Hölder continuous potential ξ.

(1) If $\nu \neq \nu_{\dim}$, then $f_\nu(\alpha)$ is real analytic and strictly convex on an interval $0 \leqslant \alpha_1 < \alpha < \alpha_2 < \infty$.

(2) For each $\alpha_1 \leqslant \alpha \leqslant \alpha_2$, there exists an equilibrium state ν_α on J such that $\nu_\alpha(K_{\nu,\alpha}) = 1$. It follows[2] that the sets $K_{\nu,\alpha}$ are dense in J.

(3) If one defines $T(q)$ by requiring that $P(-T(q)\log|a(x)| + q(\xi - P(\xi)) = 0$, then $\alpha_1 = -\lim_{q\to\infty} T'(q)$ and $\alpha_2 = -\lim_{q\to-\infty} T'(q)$.

(4) If $\nu_{\max} = \nu_{\dim}$, then $f_{\nu_{\max}}(\alpha) = \dim_H J$ for $\alpha = \dim_H J$ and $f_{\nu_{\max}}(\alpha) = 0$ for $\alpha \neq \dim_H J$.

The following theorem establishes a formula for the pointwise dimension of a measure involving the Lyapunov exponent and provides the link between the Lyapunov spectrum and the dimension spectrum. The proof of this statement with pointwise dimension replaced by Markov pointwise dimension is quite elementary. However proving this formula for actual pointwise dimension is not trivial and seems to be a new result.

Theorem 2.3. Let $g: J \to J$ be a $C^{1+\gamma}$ conformal expanding map and let $\nu = \nu_\xi$ be the equilibrium state corresponding to the Hölder continuous potential ξ. Then

$$d_\nu(x) = \frac{P(\xi) - \bar{\xi}(x)}{\chi(x)} = \frac{h_\nu(g) + \int \xi \, d\nu - \bar{\xi}(x)}{\chi(x)} \tag{2}$$

provided that $\bar{\xi}(x)$ and $\chi(x)$ exist, where

$$\bar{\xi}(x) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi(g^k(x))$$

denotes the Birkhoff average of ξ and $h_\nu(g)$ denotes the measure theoretic entropy of the map g with respect to the measure ν [W].

Proof. Let $\mathfrak{R}$ be a Markov partition for g and let μ be the pullback of ν under the coding map π. Then by the Gibbs property of μ there exist positive constants D_1 and D_2 such that for all ω

$$D_1 \exp\left(\sum_{k=0}^{n-1} \xi^*(\sigma^k(\omega)) - nP(\xi^*)\right)$$

$$\leqslant \mu(C_n(\omega)) \leqslant D_2 \exp\left(\sum_{k=0}^{n-1} \xi^*(\sigma^k(\omega)) - nP(\xi^*)\right)$$

[2] This follows since equilibrium states are fully supported on J and assign positive measure to all non-trivial open subsets.

If $x=\pi(\omega)$, then since π is uniformly bounded-to-one, we have that $P(\xi^*)=P(\xi)$ and thus

$$\lim_{n\to\infty}\frac{\log\mu(C_n(\omega))}{\log\prod_{k=0}^{n-1}|a(\pi(\sigma^k(\omega)))|^{-1}}=\frac{\bar{\xi}(x)-P(\xi)}{-\chi(x)}=\frac{P(\xi)-\bar{\xi}(x)}{\chi(x)}$$

The theorem immediately follows from Theorem 2.2. ▮

For the particular choice of potential $\xi(x)=-s\log\|dg_x\|$ we obtain the following formula for the Lyapunov exponent.

Corollary 2.1. The Lyapunov exponent

$$\chi(x)=\frac{P(-s\log|g'|)}{d_{\mu_s}(x)-s}$$

where μ_s is the equilibrium state for $-s\log\|dg_x\|$.

Letting $s=0$ we obtaining the following simple formula

Corollary 2.2. The Lyapunov exponent

$$\chi(x)=\frac{h_{\text{top}}(g)}{d_{\mu_{\max}}(x)}$$

where $h_{\text{top}}(g)$ denotes the topological entropy of the map g and $\mu_{\max}$ denotes the measure of maximal entropy.

Assume that the Lyapunov exponent $\chi(x)$ exists at a point x. For an arbitrary equilibrium state, the numerator in (2) may or may not be defined for this value of x. However, for the measure of maximal entropy, the numerator *always* exists and equals the topological entropy. Thus for *all* $x\in L_\beta$ we have that $d_{\nu_{\max}}(x)=h_{\text{top}}(g)/\beta$.

We present several applications of this remark. The first application establishes the link between the Lyapunov spectrum and the dimension spectrum for the measure of maximal entropy and is an immediate application of Theorem 2.2 and Corollary 2.2.

Theorem 2.4. Let $g: J\to J$ be a $C^{1+\gamma}$ conformal repeller. Then

(1) If $\nu_{\max}\neq\nu_{\dim}$, then the function $l(\beta)=f_{\nu_{\max}}(h_{\text{top}}(g)/\beta)$ is real analytic and strictly convex on an interval $0\leqslant\beta_1<\beta<\beta_2<\infty$.

(2) For each $\beta_1\leqslant\beta\leqslant\beta_2$, there exists an equilibrium state η_β on J such that $\eta_\beta(L_\beta)=1$. It follows that the level sets L_β are dense in J.

(3) If one defines $T(q)$ by requiring that $P(-T(q) \log |a(x)| - q h_{\text{top}}(g)) = 0$, then $\beta_1 = -\lim_{q \to \infty} T'(q)$ and $\beta_2 = -\lim_{q \to -\infty} T'(q)$.

(4) If $\nu_{\max} = \nu_{\dim}$, then $l(\beta) = d$ for $\beta = h_{\text{top}}(g)/d$ and $l(\beta) = 0$ for $\beta \neq h_{\text{top}}(g)/d$, where $d = \dim_H J$ and $h_{\text{top}}(g)$ is the topological entropy of g.

Remarks. (1) Since the sets L_β are dense, one cannot replace the Hausdorff dimension in the definition of the Lyapunov spectrum $l(\beta)$ by box dimension, for the box dimension of a set coincides with the box dimension of the closure of the set. This would lead to a trivial spectrum of dimensions.

(2) Applying Simpelaere's variational formula [Si], Schmelling [Sc] has shown that the Lyapunov spectrum $[\beta_1, \beta_2]$ is *full*, in that $\chi(x)$ attains no values $\beta \notin [\beta_1, \beta_2]$.

(3) It is an immediate consequence of the Birkhoff Ergodic Theorem that there does not exist any ergodic invariant measure which assigns positive measure to the set of points for which the Lyapunov exponent does not exist.

The next proposition follows immediately from Theorem 2.4

Proposition 2.1. Let $g: J \to J$ be a $C^{1+\gamma}$ conformal repeller for which $\nu_{\max} \neq \nu_{\dim}$. Then the range of $\chi(x)$ contains an open interval, and hence the function $\chi(x)$ attains uncountably many distinct values.

We obtain the following rigidity result as a simple corollary of Proposition 2.1.

Proposition 2.2. Let $g: J \to J$ be a $C^{1+\gamma}$ conformal repeller. If the Lyapunov exponent $\chi(x)$ attains only countable many values, then $\nu_{\max} = \nu_{\dim}$.

Combining this proposition with a theorem of Zdunik [Z], we obtain an interesting rigidity theorem for hyperbolic rational maps.

Theorem 2.5. If the Lyapunov exponent of a rational map having a hyperbolic Julia set attains only countably many values, then the map must be of the form $z \to z^{\pm n}$.

Let $g: J \to J$ be a conformal expanding map and ν an equilibrium state. In [BS] (which is an extension of previous work in [Sh]), the authors show the set of points where the pointwise dimension of ν does not exist typically has full Hausdorff dimension, i.e., the Hausdorff dimension coincides with the Hausdorff dimension of J, provided that $\nu \neq \nu_{\mathrm{dim}}$. Applying this result and Corollary 2.2 to the measure ν_{max} we obtain the following result.

Proposition 2.3. Let $g: J \to J$ be a conformal expanding map such that $\nu_{\mathrm{max}} \neq \nu_{\mathrm{dim}}$. Then the set of points for which $\chi(x)$ does not exist is dense and the Hausdorff dimension of this set coincides with the Hausdorff dimension of J.

III. AXIOM-A SURFACE DIFFEOMORPHISMS

Let M be a smooth surface and $f: M \to M$ a $C^{1+\gamma}$ diffeomorphism for $\gamma > 0$. A compact f-invariant subset $\Lambda \subset M$ is called *hyperbolic* if there exists a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$ into two subspaces and constants $C > 0$ and $0 < \lambda < 1$ such that for every $x \in \Lambda$

(1) $df E^s(x) = E^s(f(x))$, $df E^u(x) = E^u(f(x))$;

(2) for all $n \geqslant 0$

$$\|df^n v\| \leqslant C\lambda^n \|v\| \qquad \text{if} \quad v \in E^s(x)$$

$$\|df^{-n} v\| \leqslant C\lambda^n \|v\| \qquad \text{if} \quad v \in E^u(x)$$

The subspaces $E^s(x)$ and $E^u(x)$ are called *stable* and *unstable subspaces* at x, respectively. Define the continuous functions $a^s(x) = \|df \mid E^s(x)\|$ and $a^u(x) = \|df \mid E^u(x)\|$.

It is well-known (see for example, [KH]) that for every $x \in \Lambda$ one can construct one-dimensional *local stable* and *unstable local manifolds*, $W^s_{\mathrm{loc}}(x)$ and $W^u_{\mathrm{loc}}(x)$ which have the following properties:

(3) $x \in W^s_{\mathrm{loc}}(x)$, $x \in W^u_{\mathrm{loc}}(x)$;

(4) $T_x W^s_{\mathrm{loc}}(x) = E^s(x)$, $T_x W^u_{\mathrm{loc}}(x) = E^u(x)$;

(5) $f(W^s_{\mathrm{loc}}(x)) \subset W^s_{\mathrm{loc}}(f(x))$, $f^{-1}(W^u_{\mathrm{loc}}(x)) \subset W^u_{\mathrm{loc}}(f^{-1}(x))$;

(6) there exist $K > 0$ and $0 < \mu < 1$ such that for every $n \geqslant 0$

$$\rho(f^n(y), f^n(x)) \leqslant K\mu^n \rho(y, x) \qquad \text{for all} \quad y \in W^s_{\mathrm{loc}}(x)$$

and

$$\rho(f^{-n}(y), f^{-n}(x)) \leqslant K\mu^n \rho(y, x) \qquad \text{for all} \quad y \in W^u_{\mathrm{loc}}(x)$$

where ρ is the distance in M induced by the Riemannian metric.

A hyperbolic set Λ is called *locally maximal* if there exists a neighborhood U of Λ such that for any closed f-invariant subset $\Lambda' \subset U$ we have $\Lambda' \subset \Lambda$. In this case

$$\Lambda = \bigcap_{-\infty < n < \infty} f^n(U)$$

A point $x \in M$ is called *non-wandering* if for each neighborhood U of x there exists $n \geqslant 1$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of all non-wandering points of f. It is a closed f-invariant set. A diffeomorphism f is called an *Axiom-A* diffeomorphism if $\Omega(f)$ is a locally maximal hyperbolic set. If f is an Axiom-A diffeomorphism then $\Omega(f)$ can be decomposed into a finite number of disjoint closed f-invariant sets, $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_n$, such that $f \mid \Lambda_i$ is topologically transitive. Each set Λ_i is said to be a *basic set* of f. See [KH] for a more complete description. We will henceforth assume that $f: \Lambda \to \Lambda$, where Λ is a basic set.

Let ξ be a Hölder continuous function on Λ and let $\nu = \nu_\xi$ be the equilibrium state for f corresponding to ξ. We remind the reader that a finite cover $\mathcal{R} = \{R_1, ..., R_p\}$ of Λ is called a *Markov partition* for f if

(1) Each *rectangle* R_i is the closure of its interior $\mathring{R}_i$.

(2) The set $\mathring{R}_i \cap \mathring{R}_j = \emptyset$ unless $i = j$.

(3) For each $x \in \mathring{R}_i \cap f^{-1}(\mathring{R}_i)$ we have

$$f(W^s_{\text{loc}}(x) \cap R_i) \subset W^s_{\text{loc}}(f(x)) \cap R_j$$

$$f(W^u_{\text{loc}}(x) \cap R_i) \supset W^u_{\text{loc}}(f(x)) \cap R_j$$

Bowen [Bo] gave a construction of Markov partitions for Axiom-A diffeomorphisms which is an essential tool for our multifractal analysis. Let $\mathcal{R}$ be a Markov partition of Λ with transition matrix $A = (a_{i,j})$. Denote by Σ_A the set of all allowable two sided sequences of integers $(\cdots i_{-2} i_{-1} i_0 i_1 \cdots)$, i.e., $a_{i_n i_{n+1}} = 1$ for every n. We define the coding map $\pi: \Sigma_A \to \Lambda$ by

$$\pi(\omega) = x = \bigcap_{n=0}^{\infty} R_{i_{-n} \cdots i_n}, \qquad \text{where} \quad R_{i_{-n} \cdots i_n} = R^s_{i_{-n} \cdots i_0} \cap R^u_{i_0 \cdots i_n}$$

where

$$R^u_{i_0 \cdots i_n} = \bigcap_{j=0}^{n} f^{-j} R_{i_j} \qquad \text{and} \qquad R^s_{i_{-n} \cdots i_0} = \bigcap_{j=-n}^{0} f^{-j} R_{i_j}$$

This coding map π is Hölder continuous, surjective, and injective on the set of points whose orbits never intersect the boundary of any element of the Markov partition.

Let $a^s(x)$ and $a^u(x)$ be the contraction and expansion coefficients of f along the stable and unstable directions, and t^s and t^u the unique roots of Bowen's equations $P(t \log |a^s(x)|) = 0$ and $P(-t \log |a^u(x)|) = 0$). In [MM], Manning and McCluskey show that $d = \dim_H \Lambda = t^s + t^u$.

Let $R(x)$ be a rectangle in the Markov partion which contains the point x. For every $y \in R(x)$ denote by m^u_y the conditional measure on $W^u(y) \cap R(x)$ generated by the equilibrium state for $-t^u \log |a^u(x)| = 0$ and denote by m^s_y the conditional measure on $W^s(y) \cap R(x)$ generated by the equilibrium state for $t^s \log |a^s(x)|$. The reference [P2, Appendix II] contains a detailed discussion of the construction of conditional measures.

The following theorem is part of the multifractal analysis of equilibrium states on basic sets of Axiom-A surface diffeomorphisms. It was first proved by Simpelaere [Si]. In [PW1] an alternate proof was presented.

Given an invariant probability measure ν, the *dimension spectrum*, denoted by $f_\nu(\alpha)$, is defined for $\alpha \geqslant 0$ by

$$f_\nu(\alpha) = \dim_H K_{\nu, \alpha} \qquad \text{where} \quad K_{\nu, \alpha} = \{x \in \Lambda \mid d_\nu(x) = \alpha\}$$

Theorem 3.1 [Si, PW3]. Let $f: \Lambda \to \Lambda$ be a $C^{1+\gamma}$ Axiom-A surface diffeomorphism. Let $\nu = \nu_\xi$ be an equilibrium state for a Hölder continuous potential.

(1) [Y] The measure ν is exact dimensional, i.e., the pointwise dimension $d_\nu(x)$ exists for ν-almost every $x \in \Lambda$ and

$$d_\nu(x) = h_\nu(f) \left(\frac{1}{\chi_\nu^+} - \frac{1}{\chi_\nu^-} \right)$$

where $h_\nu(f)$ is the measure theoretic entropy of f and χ_ν^+, χ_ν^- are *positive and negative Lyapunov exponents of* ν, i.e.,

$$\chi_\nu^+ = \int_\Lambda a^u(x)\, d\nu(x) \qquad \text{and} \qquad \chi_\nu^- = \int_\Lambda a^s(x)\, d\nu(x)$$

(2) If $\nu \mid R(x)$ is not equivalent to $m^s \times m^u$ for any $x \in \Lambda$ (or equivalently ν is not the measure of maximal dimension) then the dimension spectrum $f_\nu(\alpha)$ is real analytic and strictly convex on an interval $0 \leqslant \alpha_1 < \alpha < \alpha_2 < \infty$.

(3) If $\nu \mid R(x)$ is equivalent to $m^s \times m^u$ for some (and thus all) $x \in \Lambda$, then $f_{\nu_{\max}}(\alpha) = \dim_H \Lambda$ for $\alpha = \dim_H \Lambda$ and $f_{\nu_{\max}}(\alpha) = 0$ for $\alpha \neq \dim_H \Lambda$.

Let Λ be a basic set for an Axiom-A surface diffeomorphism $f: M \to M$. Define the *positive* and *negative Lyapunov exponents* $\chi^+(x)$ and $\chi^-(x)$ by

$$\chi^+(x) = \lim_{n \to \infty} \frac{\log \|df^n \mid E^u(x)\|}{n} = \lim_{n \to \infty} \frac{\log \prod_{k=0}^{n-1} a^u(f^k(x))}{n}$$

and

$$\chi^-(x) = \lim_{n \to \infty} \frac{\log \|df^n \mid E^s(x)\|}{n} = \lim_{n \to \infty} \frac{\log \prod_{k=0}^{n-1} a^s(f^k(x))}{n}$$

if the limits exist. Since $df \mid E^u(x)$ is expanding and $df \mid E^s(x)$ is contracting, if the limits exist they must be non-zero. If ν is an invariant Borel probability measure, it follows from the Subadditive Ergodic Theorem that $\chi^+(x)$ and $\chi^-(x)$ exist for ν almost every x and define f-invariant measurable functions.

Let $L_\beta^+ = \{x \in \Lambda \mid \chi^+(x) = \beta\}$. Consider the following decomposition of the set Λ associated with values of the Lyapunov exponent $\chi^+(x)$ at points $x \in \Lambda$

$$\Lambda = \bigcup_{\beta \in \mathbb{R}^+} L_\beta^+ \cup \{x \in \Lambda \mid \chi^+(x) \text{ does not exist}\}$$

If ν is an ergodic measure for f, then $\chi^+(x) = \chi_\nu^+$ for ν-almost every $x \in \Lambda$. If ν is the equilibrium state corresponding to a Hölder continuous function, this set is everywhere dense. We can ask the same questions about this decomposition as we did in the case of conformal expanding maps. In a similar spirit as for conformal expanding maps we introduce the *(positive) Lyapunov spectrum* of f by

$$\ell^+(\beta) = \dim_H L_\beta^+$$

By slightly modifying the proof of Corollary 2.2 one can show that for *all* $x \in L_\beta^+$ we have that $d_{\nu_{\max}^u}(x) = h_{\text{top}}(f)/\beta$, where $d_{\nu_{\max}^u}(x)$ denotes the pointwise dimension of the conditional measure induced by $\nu_{\max}$ on the local unstable manifolds. The next theorem now follows from Theorem 3.1.

Theorem 3.2. Let $f: \Lambda \to \Lambda$ be a $C^{1+\gamma}$ Axiom-A surface diffeomorphism.

(1) If $\nu_{\max}^u \mid R(x)$ is not equivalent to $m^u \mid R(x)$ for any $x \in \Lambda$, then for any x the Lyapunov spectrum $\ell^+(\beta) = f_{\nu_{\max}^u}(h_{\mathrm{top}}(g)/\beta)$ is real analytic and strictly convex on an interval $0 \leqslant \beta_1 < \beta < \beta_2 < \infty$.

(2) For each $\beta_1 < \beta < \beta_2$, the set L_β^+ is dense in Λ.

(3) If $\nu_{\max}^u \mid R(x)$ is equivalent to $m^u \mid R(x)$ for some (and hence all) $x \in \Lambda$, then $\ell^+(\beta) = \dim_H \Lambda$ for $\beta = h_{\mathrm{top}}(f)/\dim_H \Lambda$ and $\ell^+(\beta) = 0$ for $\beta \neq h_{\mathrm{top}}(f)/\dim_H \Lambda$.

As immediate consequences of Theorem 3.2 we obtain the analogs of the consequences of Theorem 2.4.

Proposition 3.1. Let $f: \Lambda \to \Lambda$ be a $C^{1+\gamma}$ Axiom-A surface diffeomorphism for which the measure $\nu_{\max_x}^u \mid R(x)$ is not equivalent to the measure $m_x^u \mid R(x)$ for any $x \in \Lambda$. Then the range of $\chi^+(x)$ contains an open interval, and hence the Lyapunov exponent $\chi^+(x)$ attains uncountably many distinct values.

We obtain the following rigidity result as a simple corollary of Proposition 3.1.

Proposition 3.2. Let $f: \Lambda \to \Lambda$ be a $C^{1+\gamma}$ Axiom-A surface diffeomorphism. If the Lyapunov exponent $\chi^+(x)$ attains only countably many values, then $\nu_{\max_x}^u \mid R(x)$ is equivalent to $m_x^u \mid R(x)$ for all $x \in \Lambda$.

Let $f: \Lambda \to \Lambda$ be a $C^{1+\gamma}$ Axiom-A surface diffeomorphism and ν an equilibrium state for a Hölder continuous potential. In [BS], the authors show that the set of points where the pointwise dimension of ν does not exist typically has full Hausdorff dimension, i.e., the Hausdorff dimension coincides with the Hausdorff dimension of Λ. Applying this result to the measure $\nu_{\max}$ and recalling that $d_{\nu_{\max_x}^u}(x) = h_{\mathrm{top}}(g)/\chi^+(x)$ for *all* $x \in \Lambda$, we obtain the following result.

Proposition 3.3. Let $f: \Lambda \to \Lambda$ be a $C^{1+\gamma}$ Axiom-A surface diffeomorphism such that $\nu_{\max_x}^u$ is not equivalent to m_x^u for any x. Then the set of points for which $\chi^+(x)$ does not exist is dense and the Hausdorff dimension coincides with the Hausdorff dimension of the basic set Λ.

Similar statements hold true for the *negative Lyapunov spectrum* of f corresponding to negative values of the Lyapunov exponent $\chi^-(x)$ at points $x \in \Lambda$. Also, as for conformal repellers, it follows from [Sc] that the Lyapunov exponents attain no other values than those which arise in Theorem 2.4.

IV. *LYAPUNOV RIGIDITY* FOR GEODESIC FLOWS ON NEGATIVELY CURVED SURFACES

Geodesic flows constitute an important class of dynamical systems. For instance, it follows from the Maupertuis Principle that every holonomic mechanical system reduces to a geodesic flow. There is a wonderful interplay between the geometry of the manifold and the dynamics of the geodesic flow. Some results say a Riemannian manifold for which the metric satisfies certain curvature estimates (e.g., everywhere negative sectional curvatures) possesses a geodesic flow with stong stochastic properties (ergodic and Bernoulli). Another more recent class of results, so called rigidity results, say that certain strong dynamical conditions for the geodesic flow on a Riemannian manifold (e.g., coincidence of certain entopies) sometimes implies strong geometric consequences for the underlying Riemannian metric (e.g., the metric is locally symmetric).

It is easy to establish a geometric analog of the (rigidity) Theorem 2.5 for geodesic flows on compact negatively curved surfaces. Let (M^2, g) be a compact negatively surface (the Gaussian curvature is negative at every point). The geodesic flow is a flow on the unit tangent bundle SM^2 of M^2 and can be described as follows: the transformation at time t carries the unit vector v located at the point x to the unit vector along the geodesic eminating from x in the direction v with footpoint having distance t from x. Alternatively, the geodesic flow is the Lagrangian flow restricted to the unit tangent bundle for the Lagrangian $\mathscr{L}(x, v) = \frac{1}{2} g_x(v, v)$. There is a natural volume element λ (Liouville measure) on the unit tangent bundle and the geodesic flow preserves it.

Let $\chi^+(v)$ denote the positive Lyapunov exponent of the geodesic flow at the vector v. Since the geodesic flow on a negatively curved surface is ergodic with respect to the Liouville measure λ [H, Ba], it follows that $\chi^+(v) = \chi^+_\lambda$ for λ-almost every v. We will show, in particular, that if $\chi^+(v)$ attains only countably many values then the metric must have *constant* negative curvature.

Theorem 4.1. Consider the geodesic flow on a negatively curved surface. If the set of vectors such that $\chi^+(v) \neq \chi^+_\lambda$ has measure zero with respect to the measure of maximal entropy $\nu_{\max}$ for the geodesic flow, then the metric must have constant curvature.

Proof. Applying the Ruelle Entropy Inequality [R3] and the Pesin Entropy Formula [P1] we have that

$$h_{\text{top}}(g) \leqslant \int_{SM^2} \chi^+(v)\, d\nu_{\max}(v) = \int_{SM^2} \chi^+(v)\, d\lambda(v) = h_\lambda(g)$$

where $h_{\text{top}}(g)$ denotes the topological entropy and $h_\lambda(g)$ denotes the Liouville entropy of the time-1 map of the geodesic flow. It then follows from Katok's Entropy Rigidity theorem [Ka] that the metric g must have constant negative curvature. This proves the theorem. ▮

As a special case we obtain the following corollary.

Corollary 4.1. Consider the geodesic flow on a negatively curved surface. If the set of vectors such that $\chi^+(v) \neq \chi_\lambda^+$ is countable, then the metric must have constant curvature.

APPENDIX

This Appendix contains essential definitions and facts from thermodynamic formalism. For details consult [Bo, R2, W]. Let X denote a compact metric space and let $C(X)$ denote the space of real valued continuous functions on X.

1. Let $g: X \to X$ be a continuous map. We define the *pressure* function $P: C(X) \to \mathbb{R}$ by

$$P(\phi) = \sup_{\mu \in \mathfrak{M}(X)} \left(h_\mu(g) + \int_X \phi \, d\mu \right)$$

where $\mathfrak{M}(X)$ denotes the set of g-invariant probability measures on X and $h_\mu(g)$ denotes the measure theoretic entropy of the map g with respect to the measure μ. A Borel probability measure $\mu = \mu_\phi$ on X is called an *equilibrium state* for the potential $\phi \in C(X)$ if

$$P(\phi) = h_\mu(g) + \int_X \phi \, d\mu$$

3. Let X and Y be compact metric spaces and suppose $\psi: X \to Y$ is a continuous surjection such that the cardinality of $\psi^{-1}(y)$ is uniformly bounded for all $y \in Y$. Then for any $\phi \in C(Y)$ we have that $P_Y(\phi) = P_X(\phi \circ \psi)$.

4. Let $\phi \in C(\Sigma_A^+)$. A Borel probability measure $\mu = \mu_\phi$ on Σ_A^+ is called a *Gibbs measure* for the potential ϕ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \leqslant \frac{\mu\{\kappa : \kappa_i = \omega_i,\ i = 0,..., n-1\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k \omega))} \leqslant D_2$$

for all $\omega = (\omega_0 \omega_1 \cdots) \in \Sigma_A^+$ and $n \geq 1$. There are similar definitions for Σ_A and Σ_A^-.

5. For subshifts of finite type Σ_A^+, Σ_A^- and Σ_A, Gibbs measures exist for any Hölder continuous potential ϕ, are unique, and coincide with the equilibrium state for ϕ.

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